Propagation of Electromagnetic Waves

Maxwell's equations were cobbled together from a **variety** of results from different scientists (e.g. Ampere, Faraday), whose work mainly was done using either **static** or **slowly** time-varying sources and fields.

Maxwell brought these results together to form a **complete** theory of electromagnetics—a theory that then predicted a most **startling** result!

To see this result, consider first the **free-space** Maxwell's Equations in a **source-free** region (e.g., a vacuum). In other words, the fields in a region **far away** from the current and charges that created them:

$$\nabla \times \mathbf{B}(\bar{r},t) = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}(\bar{r},t)}{\partial t}$$

$$\nabla \times \mathbf{E}(\bar{r},t) = -\frac{\partial \mathbf{B}(\bar{r},t)}{\partial t}$$

$$\nabla \cdot \mathbf{E}(\bar{r},t) = 0$$

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Say we take the curl of Faraday's Law:

$$\nabla \times \nabla \times \mathbf{E}(\bar{r},t) = -\frac{\partial \nabla \times \mathbf{B}(\bar{r},t)}{\partial t}$$

Inserting Ampere's Law into this, we get:

$$\begin{aligned}\nabla \times \nabla \times \mathbf{E}(\bar{r},t) &= -\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \mathbf{E}(\bar{r},t)}{\partial t}\right)\\ &= -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}(\bar{r},t)}{\partial t^2}\end{aligned}$$

Recalling that if $\nabla \cdot \mathbf{E}(\bar{r}) = 0$ then $\nabla \times \nabla \times \mathbf{E}(\bar{r}) = \nabla^2 \mathbf{E}(\bar{r})$, we can write the following differential equation, one which describes the **behavior** on an electric field in a **vacuum**:

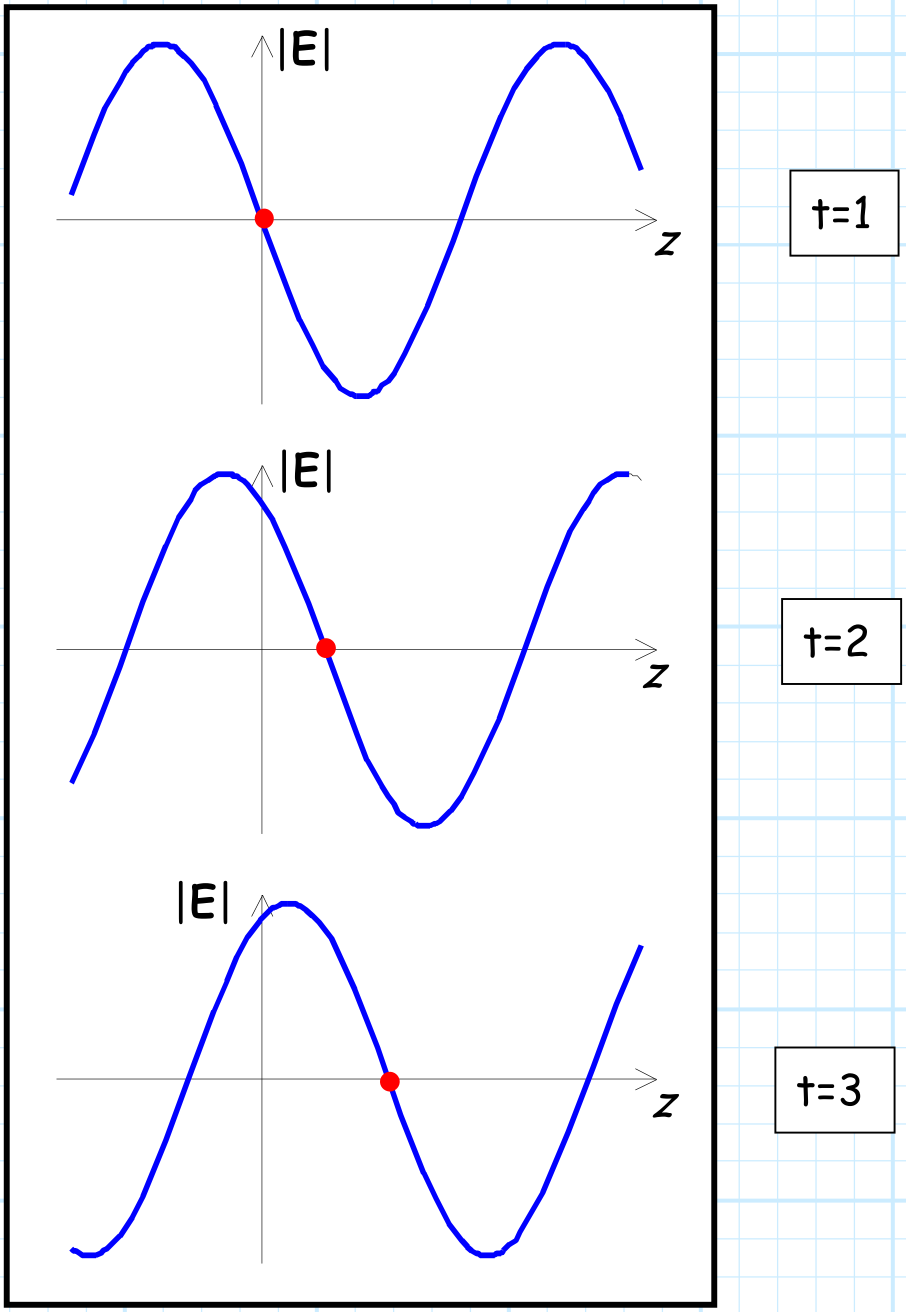
$$\nabla^2 \mathbf{E}(\bar{r},t) + \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}(\bar{r},t)}{\partial t^2} = 0$$

Among the **many** solutions to this differential equation is:

$$\mathbf{E}(\bar{r},t) = \hat{a}_e \sin \omega\left(t - z\sqrt{\mu_0 \varepsilon_0}\right)$$

In this case, the electric field is varying with time in a **sinusoidal** manner, with an angular **frequency** of ω radians/sec. The direction of the electric field is orthogonal to the z-axis (i.e., $\hat{a}_e \cdot \hat{a}_z = 0$).

Lets plot this function at **three different times**:



Here the red dot indicates a "phase" of zero, i.e., $\phi = \omega\left(t - z\sqrt{\mu_0\varepsilon_0}\right) = 0$. Note that this dot appears to be **moving forward** along the *z*- axis as a function of time.

➡ **The electric field is moving !**

Q: *How **fast** is it moving?*

A: Lets see how fast the **red dot** is moving! Rearranging $\omega\left(t - z\sqrt{\mu_0\varepsilon_0}\right) = 0$, we get the position z of the dot as a function of time t:

$$z = \frac{t}{\sqrt{\mu_0\varepsilon_0}}$$

Its **velocity** is just the **time derivative** of its position:

$$v_p = \frac{dz}{dt} = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

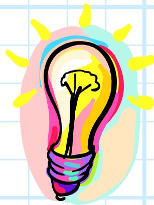
Hey we can calculate this! The **electric field** is moving at a velocity of:

$$\begin{aligned} v_p &= \frac{1}{\sqrt{\mu_0\varepsilon_0}} \\ &= \frac{1}{\sqrt{\left(4\pi x10^{-7}\right)\left(8.854x10^{-12}\right)}} \\ &= \mathbf{3x10^8} \quad \left[\frac{\text{meters}}{\text{second}}\right] \end{aligned}$$

Q: *Hey wait minute! 3 x 10^8 meters/second—that's the **speed of light**!?!*

A: True! We find that the magnetic field will likewise move in the **same** direction and with the **same** velocity as the electric field.

We call the combination of the two fields a **propagating** (i.e., moving) **electromagnetic wave**.



Light is a propagating electromagnetic wave!

This was a **stunning** result in Maxwell's time. No one had linked light with the phenomena of electricity and magnetism. Among other things, it meant that "light" could be made with much greater wavelengths (i.e., **lower frequencies**) than the light visible to us humans.

Henrich Hertz first succeeded in creating and measuring this low frequency "light". Since then, humans have put this low-frequency light to **great** use. We often refer to it as a "**radio waves**"—a propagating electromagnetic wave with a frequency in the range of 1 MHz to 20 GHz. We use it for all "**wireless**" technologies !